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Analysis of the effects of time delay in nonlinear systems using generalised frequency response functions

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Abstract

A recursive algorithm for computing the generalised frequency response functions (GFRFs) of nonlinear time delay systems described by nonlinear differential-difference equation models is derived using the operator ε_n and used to analyse the effects of time delay in nonlinear systems. The algorithm shows an explicit relationship between the model parameters and the GFRFs. Such a relationship provides important insight into the behaviour of nonlinear systems. The effect of delay on several properties of nonlinear systems such as *harmonic generation, gain compression/expansion* and *desensitisation* is studied by considering the example of a Duffing oscillator with retarded damping. Results of the studies convincingly demonstrate that the system delay has a significant effect on several important properties of nonlinear systems.

1. Introduction

Identification of nonlinear systems using the Volterra and Wiener kernels have received considerable attention in the past several years [1–6]. Based on the Volterra theory, the nonlinear system is characterised either by the Volterra kernels in the time domain or equivalently by the transformation of the Volterra kernel into the frequency domain that is called generalised frequency response functions (GFRFs).

The GFRFs can be computed directly from the input–output data [7–9]. An alternative approach is to estimate a parametric model of the system and subsequently derive the GFRFs from this model using harmonic probing techniques [10–14].

Although all these techniques and approaches have been applied to map systems modelled by either differential equations or NARX models [15], the frequency response functions (FRFs) of nonlinear time delay systems have not received much attention in the last several years except for preliminary analysis of linear time delay systems in the frequency domain [16]. But many physical systems in the field of aeronautics, bioscience,

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chemical process control, economics, feed back control, distributed computing, etc. [17–26] are governed by differential equations with retarded arguments, i.e. functional differential equations or differential difference equations, and are called time delay systems. A wide class of complicated systems like the distributed parameter system and those governed by parabolic or hyperbolic differential equations can reasonably be modelled by differential equations with distributed time delays. Examples from applications in which delays are important have been discussed in Ref. [27] from the perspective of optimal control.

Systems described by retarded differential-difference equations are essentially a class of infinite-dimensional systems and their analysis is computationally much involved. Although some advances have been made in understanding the relations between infinite-dimensional dynamical systems and finite-dimensional ones [28], understanding the physical behaviour of these systems in the time domain is not without difficulties. It is therefore necessary to analyse these systems in the frequency domain to exploit the obvious advantages of such a domain.

The purpose of the present study is three fold: (i) to derive expressions for the GFRFs for a class of nonlinear time delay systems described by nonlinear differential-difference equations using the operator ε_n introduced by Zhang et al. [13], (ii) to show the effects of different types of nonlinear terms, e.g. pure input, pure output and input–output cross-product terms on GFRFs and (iii) to investigate the effects of delay on the nonlinear phenomena of harmonics, gain compression and expansion and desensitisation.

The organisation of the paper proceeds as follows: Section 2 briefly reviews the Volterra modelling of nonlinear systems and explains the concept of computing GFRF for a nonlinear system by considering an example of a nonlinear delay system. The general form of nonlinear differential-difference equation models is presented in Section 3 and a relationship that maps the parameters of these models directly into the GFRFs is derived. In Section 4, the effects of delay on harmonic generation, gain compression/expansion and desensitisation has been illustrated with the example of a nonlinear Duffing oscillator with retarded damping with conclusions in Section 5.

2. Background

In this section, a brief review of the background material necessary to understand the results of the paper is given.

2.1. Volterra modelling of single-input single-output (SISO) systems

Consider a nonlinear system whose output can be described as

$$y(t)=\sum_{n=1}^{N_l} y_n(t), \tag{1}$$

where N_l is the maximum degree of nonlinearity, and $y_n(t)$, the nth-order output of the system, is given by

$$y_n(t)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n)\,\mathrm{d}\tau_1,\ldots,\mathrm{d}\tau_n\prod_{i=1}^{n} u(t-\tau_i)\,\mathrm{d}\tau_i,\quad n>0, \tag{2}$$

where $h_n(\tau_1,\ldots,\tau_n)$ is the nth-order Volterra kernel [29]. Volterra generalised the linear convolution concept to deal with nonlinear systems by replacing the single impulse response with a series of multidimensional integration kernels. The nth-order Volterra kernel describes nonlinear interactions among n copies of the input. The multidimensional Fourier transform of the nth-order Volterra kernel gives the nth-order transfer function or GFRF

$$H_n(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n)\mathrm{e}^{-\mathrm{j}(\omega_1\tau_1+\cdots+\omega_n\tau_n)}\,\mathrm{d}\tau_1,\ldots,\mathrm{d}\tau_n. \tag{3}$$

The nth-order kernel and the kernel transform are not necessarily unique because an interchange of arguments in $h_n(\tau_1,\ldots,\tau_n)$ may give different kernels without affecting the input–output relationships. To ensure that the

GFRFs are unique, these are symmetrised to give

$$H_n^{\text{sym}}(j\omega_1, \ldots, j\omega_n) = \frac{1}{n!} \sum_{\substack{\text{all permutations} \\ \omega_1,\ldots,\omega_n}} H_n(j\omega_1, \ldots, j\omega_n). \tag{4}$$

2.2. Computation of GFRF for SISO systems

The computation of the GFRFs from the parametric model of the system has been reported by Billings and co-workers in the past [11,12,30]. The procedure to extract nth-order GFRF introducing an extraction operator ε_n have been derived for SISO nonlinear systems by Zhang et al. [13] and for nonlinear MIMO systems by Swain and Billings [14], Worden et al. [31]. This will be used in the derivation of GFRF for time delay systems. The reader should be aware that in Ref. [30, Section 8, p. 876], an expression for GFRF of time delay nonlinear systems is given; however, it is appropriate to mention that the present study derives the FRFs for time delay nonlinear systems using the operator ε_n.

For the better understanding of the work, the operator ε_n is now briefly commented. Before deriving the expressions for the GFRF, it is appropriate to highlight the assumptions involved in the derivation. Note that the GFRFs are related to the Volterra kernels by multidimensional Fourier transforms. Volterra series is an infinite series and its convergence over all ranges of input excitation has been a measure concern. Some of the early and recent results have partially established bounds on the radius of convergence [32–34]. For the systems under study, it is assumed that the Volterra series converges over the range of input excitations of interest. Further it is assumed that all the systems considered in the present study can be represented by the Volterra series. Palm and Poggio [35] have derived the necessary and sufficient conditions for the existence of the Volterra series for a given class of systems. Their results essentially show that for any system where the nonlinearity is analytic (e.g. the polynomial-type nonlinearity), or where the nonlinearity can be approximated with an arbitrary accuracy by polynomial systems by the Stone and Weierstrass theorem [36] can be represented by Volterra series. Since the present study consider systems described by nonlinear differential equations where the nature of the nonlinearity is polynomial, it satisfies the conditions postulated by Palm and Poggio [35]. Note that introduction of delay in the output or the input of the differential equation does not affect the analytic property of the system.

Consider a system where a parametric model is assumed to exist and is represented as

$$M(t; \theta, y, u) = 0, \tag{5}$$

where $M(\cdot)$ is a functional of the input u, output y and θ is a set of model parameters. As y in Eq. (5) can be expressed in terms of H and u, this can be written as

$$M(t; \theta, H, u) = 0. \tag{6}$$

Computation of $H(\cdot)$ by manipulating Eq. (6) for arbitrary inputs often produces complicated integral equations. However, the harmonic probing technique [10,11] can be used to compute H from Eq. (5). This involves applying an input consisting of R complex exponentials of frequency ω_σ defined as

$$u(t) = \sum_{\sigma=1}^{R} e^{j\omega_\sigma t}. \tag{7}$$

The spectrum of the input is

$$U(j\omega) = \sum_{\sigma=1}^{R} 2\pi\delta(j\omega - j\omega_\sigma), \tag{8}$$

where δ is the so-called Dirac delta-function.

The output of the system under the harmonic excitation of Eq. (7) becomes [13]

$$y(t) = \sum_{n=1}^{N_l} \sum_{\substack{\text{[all perm. of } R \text{ freq.} \\ \text{taken } n \text{ at a time]}}} \sum_{\substack{\text{[all perm. of} \\ \omega_{\sigma_1},\ldots,\omega_{\sigma_n}]}} H_n(\mathrm{j}\omega_{\sigma_1},\ldots,\mathrm{j}\omega_{\sigma_n})\mathrm{e}^{\mathrm{j}(\omega_{\sigma_1}+\cdots+\omega_{\sigma_n})t}. \tag{9}$$

To find the nth-order GFRF, $H_n(\cdot)$, it is convenient to consider the special case $R = n$, so that there is only one non-repetitive combination of frequencies $\{\omega_1,\ldots,\omega_n\}$ among all the possibilities.

Substituting Eqs. (9) and (7) into Eq. (6) yields the following equation:

$$M(t;\theta,H,\omega_\sigma) = 0, \tag{10}$$

where ω_σ includes the frequencies $\{\omega_1,\ldots,\omega_R\}$. To compute $H_n(\cdot)$, R is made equal to $n.M(\cdot)$ will contain many exponential terms but we are only interested in the term with non-repetitive frequencies $\mathrm{e}^{\mathrm{j}(\omega_1+\cdots+\omega_n)t}$.

For a given expression, the operator $\varepsilon_n[\cdot]$ for SISO systems involves the execution of the following steps:

(i) Substitute the harmonic input of Eq. (7) and corresponding Volterra expansion of the output (Eq. (9)) into the given expression.
(ii) Express the output $y(t)$ as a function of H and ω_σ.
(iii) Extract the coefficient of $\mathrm{e}^{\mathrm{j}(\omega_1+\omega_2+\cdots\omega_n)t}$ from the resulting expression.

Before deriving the general expression for GFRF of an nth-order nonlinear time delay system, computation of GFRF using harmonic probing technique is illustrated through an example.

Example 1. Consider a nonlinear system with retarded damping described as follows [37]:

$$\ddot{y}(t) + a_1\dot{y}(t-T_1) + a_2 y(t) + c_1\dot{y}^2(t-T_2) = b_1 u(t). \tag{11}$$

To compute $H_1(\mathrm{j}\omega_1)$, the one-tone input $\mathrm{e}^{\mathrm{j}\omega_1 t}$ is applied. The output is given by

$$y(t) = H_1(\mathrm{j}\omega_1)\mathrm{e}^{\mathrm{j}\omega_1 t}, \quad u(t) = \mathrm{e}^{\mathrm{j}\omega_1 t}. \tag{12}$$

By substituting the values of $y(t)$ and $u(t)$ in Eq. (11) and comparing the coefficients of $\mathrm{e}^{\mathrm{j}\omega_1 t}$ we get

$$[(\mathrm{j}\omega_1)^2 + a_1(\mathrm{j}\omega_1)\mathrm{e}^{-\mathrm{j}\omega_1 T_1} + a_2]H_1(\mathrm{j}\omega_1) = b_1.$$

Thus

$$H_1(\mathrm{j}\omega_1) = \frac{b_1}{(\mathrm{j}\omega_1)^2 + a_1(\mathrm{j}\omega_1)\mathrm{e}^{-\mathrm{j}\omega_1 T_1} + a_2}. \tag{13}$$

To compute $H_2(\mathrm{j}\omega_1,\mathrm{j}\omega_2)$, a two-tone input

$$u(t) = \mathrm{e}^{\mathrm{j}\omega_1 t} + \mathrm{e}^{\mathrm{j}\omega_2 t} \tag{14}$$

is applied to the system to give the output

$$y(t) = H_1(\mathrm{j}\omega_1)\mathrm{e}^{\mathrm{j}\omega_1 t} + H_1(\mathrm{j}\omega_2)\mathrm{e}^{\mathrm{j}\omega_2 t} + 2!H_2^{\mathrm{sym}}(\mathrm{j}\omega_1,\mathrm{j}\omega_2)\mathrm{e}^{\mathrm{j}(\omega_1+\omega_2)t} + \text{terms of repetitious combinations of frequencies}. \tag{15}$$

Now by substituting the values of $y(t)$ and $u(t)$ in Eq. (11) and extracting the coefficients of $\mathrm{e}^{\mathrm{j}(\omega_1+\omega_2)t}$ we get

$$\begin{aligned}&[(\mathrm{j}\omega_1+\mathrm{j}\omega_2)^2 + a_1(\mathrm{j}\omega_1+\mathrm{j}\omega_2)\mathrm{e}^{-\mathrm{j}(\omega_1+\omega_2)T_1} + a_2]2!H_2^{\mathrm{sym}}(\mathrm{j}\omega_1,\mathrm{j}\omega_2)\\ &\quad = -c_1[(\mathrm{j}\omega_1)H_1(\mathrm{j}\omega_1)\mathrm{e}^{-\mathrm{j}\omega_1 T_2}(\mathrm{j}\omega_2)H_1(\mathrm{j}\omega_2)\mathrm{e}^{-\mathrm{j}\omega_2 T_2}] - c_1(\mathrm{j}\omega_2)H_1(\mathrm{j}\omega_2)\mathrm{e}^{-\mathrm{j}\omega_2 T_2}(\mathrm{j}\omega_1)H_1(\mathrm{j}\omega_1)\mathrm{e}^{-\mathrm{j}\omega_1 T_2}.\end{aligned} \tag{16}$$

From Eq. (16) $H_2^{\mathrm{sym}}(\cdot)$ can easily be derived.

3. Generalised frequency response functions for delay differential equation models for nonlinear time delay systems

A wide class of nonlinear time delay system can be described by the nonlinear delay differential equation of the form

$$\sum_{n=1}^{N_l}\sum_{p=0}^{n}\sum_{l_1,l_{p+q}=0}^{L} c_{pq}(l_1,\ldots,l_{p+q}:T_{l_1},\ldots,T_{l_{p+q}})\prod_{i=1}^{p}D^{l_i}y(t-T_{l_i})\prod_{i=p+1}^{p+q}D^{l_i}u(t-T_{l_i})=0, \tag{17}$$

where $p+q=n$ and the operator D^{l_i} is defined as

$$D^{l_i}x(t)=\frac{\mathrm{d}^{l_i}}{\mathrm{d}t^{l_i}}. \tag{18}$$

L is the order of maximum differential. The parameter $c_{p,q}(l_1,\ldots,l_{p+q}:T_{l_1},\ldots,T_{l_{p+q}})$ is associated with the term of the form $\prod_{i=1}^{p}D^{l_i}y(t-T_{l_i})\prod_{i=p+1}^{p+q}D^{l_i}u(t-T_{l_i})$. $T_{l_1},\ldots,T_{l_{p+q}}$ are delays of the input–outputs. Note that the notations used in the present representation are similar to those used in Refs. [30,38].

Example 2. As an example the delay differential equation

$$\begin{aligned}&\ddot{y}(t-0.5)+3\dot{y}(t-0.1)+4.2y(t-0.3)+3.2\dot{y}(t-2)y^2(t-5)\\&\quad+5.1y(t-0.6)\dot{u}(t-0.7)+\dot{u}(t-0.1)u(t-0.7)=0\end{aligned} \tag{19}$$

would be represented in the above form as

$$\begin{aligned}&c_{10}(2:0.5)=1.0,\\&c_{10}(1:0.1)=3.0,\\&c_{10}(0:0.3)=4.2,\\&c_{30}(1,0,0:2,5,5)=3.2,\\&c_{11}(0,1:0.6,0.7)=5.1,\\&c_{02}(1,2:0.1,0.7)=1.0.\end{aligned}$$

It can be noticed that model (17), and consequently, model (19) consist of various terms that can be divided into three types: pure inputs, pure outputs and input–output cross-product terms. Although applying the extraction operator ε_n to the system of Eq. (17) will yield an algebraic equation whose solution gives the GFRFs, it will be physically more appealing if the contribution of each type of nonlinearity on the GFRF can be explicitly expressed independent of other nonlinear terms. These will be described in the following remarks.

Remark 1. Pure input nonlinear terms.

While computing $H_n(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)$, the effect of applying ε_n operator to a pure input nonlinear term denoted as $[U^N]$ is given by

$$\begin{aligned}\varepsilon_n[U^N]&=\sum_{\substack{\text{all permutations of}\\ \omega_1,\ldots,\omega_n}}(\mathrm{j}\omega_1)^{l_1}\cdots(\mathrm{j}\omega_n)^{l_n}\mathrm{e}^{-\mathrm{j}(\omega_1T_{l_1}+\cdots+\omega_nT_{l_n})}\quad\text{for } n=N\\&=0\quad\text{otherwise,}\end{aligned} \tag{20}$$

where $[U^N]$ is the nth-order nonlinear terms of the input that are of the form $\prod_{i=1}^{n}D^{l_i}u(t-T_{l_i})$.

Remark 2. Pure output nonlinear terms.

A pure output nonlinear term of degree p will contribute to the nth-order GFRF when $p\leqslant n$. Applying ε_n operator to the pth-order factor of pure output term denoted as $[Y^P]$, which are of the form $\prod_{i=1}^{p}D^{l_i}y(t-T_{l_i})$, is given by

$$\varepsilon_n[Y^p] = \sum_{\substack{\text{all permutations of} \\ \omega_1,\ldots,\omega_n}} H_{np}^{\text{asym}}(j\omega_1,\ldots,j\omega_n) \quad \text{for } p \leqslant n$$
$$= 0 \quad \text{for } p > n, \tag{21}$$

where

$$H_{np}^{\text{asym}}(j\omega_1,\ldots,j\omega_n) = \sum_{i=1}^{n-p+1} H_i(j\omega_1,\ldots,j\omega_i)H_{n-i,\,p-1}(j\omega_{i+1},\ldots,j\omega_n)(j\omega_1+\cdots+j\omega_i)^{l_p}\mathrm{e}^{-j(\omega_1+\cdots+\omega_i)T_{l_p}} \tag{22}$$

denotes the contribution of the pth-order factor of nonlinear output term to the nth-order nonlinearity. This is estimated recursively and the recursion finishes with $p=1$, with $H_{n,1}(j\omega_1,\ldots,j\omega_n)$ having the property

$$H_{n,1}(j\omega_1,\ldots,j\omega_n) = H_n(j\omega_1,\ldots,j\omega_n)(j\omega_1+\cdots+j\omega_n)^{l_1}\mathrm{e}^{-j(\omega_1+\cdots+\omega_n)T_{l_1}}. \tag{23}$$

Remark 3. Input–output cross-product terms.

The pth-order factor of the output Y^p in conjunction with the qth-order factor of the input $[U^q]$ will contribute to the nth-order GFRF provided $p+q \leqslant n$. By applying the extraction operator ε_n to $[Y^pU^q]$, we get

$$\varepsilon_n[Y^pU^q] = \sum_{q=1}^{n-1}\sum_{l_1,l_{p+q}=0}^{L}(j\omega_{n-q+1})^{l_{p+1}},\ldots,(j\omega_n)^{l_{p+q}}\mathrm{e}^{-j(\omega_{n-q+1}T_{l_{p+1}}+\cdots+\omega_nT_{l_{p+q}})}$$
$$\times H_{n-q,p}(j\omega_1,\ldots,j\omega_{n-q}) \quad \text{for } p+q \leqslant n$$
$$= 0 \quad \text{otherwise.} \tag{24}$$

The terms that contribute are of the form $\prod_{i=1}^{p}D^{l_i}y(t-T_{l_i})\prod_{i=p+1}^{p+q}D^{l_i}u(t-T_{l_i})$ and the associated coefficient is $c_{pq}(l_1,\ldots,l_{p+q}:(T_{l_1},\ldots,T_{l_{p+q}})$.

From the Remarks 1, 2 and 3, it is obvious that among all the terms present in Eq. (17) only the linear output terms will produce a term $\mathrm{e}^{j(\omega_1+\cdots+\omega_n)t}$ with $H_n(j\omega_1,\ldots,j\omega_n)$ appearing as a coefficient. All other terms will only produce terms with lower order $H_i(\cdot),\ i<n$ as the coefficients. Also, for the valid input–output map to exist, it is essential that there must be at least one linear output term present in the system model. Since applying ε_n operator to linear output terms produces a term with coefficient $H_n(\cdot)$, the contribution of linear output terms from Eq. (17) are brought to the left-hand side and all other terms are taken to the right-hand side. Thus, the mapping of Eq. (17) gives

$$-\left[\sum_{l_1=0}^{L}c_{10}(l_1,:T_{l_1})(j\omega_1+\cdots+j\omega_n)^{l_1}\mathrm{e}^{-j(\omega_1+\cdots+\omega_n)T_{l_1}}\right]n!H_n(j\omega_1,\ldots,j\omega_n)$$
$$= \sum_{l_1,l_n=0}^{L}c_{0,n}(l_1,\ldots,l_n:T_{l_1},\ldots,T_{l_n})\varepsilon_n[U^n]\sum_{p=2}^{n}\sum_{l_1,l_p}^{n}c_{p,0}(l_1,\ldots,l_p:T_{l_1},\ldots,T_{l_p})\varepsilon_n[Y^p]$$
$$\times\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{l_1,l_{p+q}=0}^{L}c_{p,q}(l_1,\ldots,l_{p+q}:T_{l_1},\ldots,T_{l_{p=q}})\varepsilon_n[Y^pU^q]. \tag{25}$$

The computation of GFRFs using the procedure described are illustrated with the following example.

Example 3. Consider a modified Vander Pol equation with retarded damping described by the equation

$$\ddot{y}(t) + 2\zeta\omega_n(1-y^2(t))\dot{y}(t-T_1) + \omega_n^2y(t) = u(t). \tag{26}$$

This system when represented in the general notations introduced in Eq. (17) gives $c_{10}(2:0)=1.0$, $c_{01}(0:0)=-1.0$, $c_{10}(1:T_1)=2\zeta\omega_n$, $c_{30}(0,0,1:0,0,T_1)=-2\zeta\omega_n$ and $c_{10}(0:0)=\omega_n^2$.

Thus, putting these values in the general expression of Eq. (25), we get

$$H_1(\mathrm{j}\omega_1) = \frac{1.0}{(\mathrm{j}\omega_1)^2 + 2\zeta\omega_n(\mathrm{j}\omega_1)\mathrm{e}^{-\mathrm{j}\omega_1 T_1} + \omega_n^2}. \tag{27}$$

Since this system does not possess any second-order nonlinear terms, $H_2(\mathrm{j}\omega_1, \mathrm{j}\omega_2)$ is absent. The third-order FRF is given by

$$\begin{aligned}
&[(\mathrm{j}\omega_1 + \mathrm{j}\omega_2 + \mathrm{j}\omega_3)^2 + 2\zeta\omega_n(\mathrm{j}\omega_1 + \mathrm{j}\omega_2 + \mathrm{j}\omega_3)\mathrm{e}^{-\mathrm{j}(\omega_1+\omega_2+\omega_3)T_1} + \omega_n^2]3!H_3(\mathrm{j}\omega_1, \mathrm{j}\omega_2, \mathrm{j}\omega_3) \\
&= 2\zeta\omega_n \sum_{\substack{\text{all permutations of} \\ \omega_1,\ldots,\omega_3}} H_{33}(\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_3) \\
&= 2\zeta\omega_n \sum_{\substack{\text{all permutations of} \\ \omega_1,\ldots,\omega_3}} H_1(\mathrm{j}\omega_1)H_1(\mathrm{j}\omega_2)H_1(\mathrm{j}\omega_3)(\mathrm{j}\omega_1)^1\mathrm{e}^{-\mathrm{j}\omega_1 T_1}.
\end{aligned} \tag{28}$$

4. Analysis of nonlinear systems with delay in frequency domain

Analysis of nonlinear systems without delay in using frequency domain techniques have been carried out by Tomlinson, Worden, Billings and co-workers [39–43]. Nonlinear systems exhibit a variety of exciting behaviour such as generation of *harmonics* and *inter modulation* frequencies together with effects of *gain compression/expansion* and *desensitisation*, which are not present in linear time invariant systems [11,44]. Before the effects of delay on some of these properties are illustrated by a specific example, it is appropriate to briefly review the related theory following Wiener and Spina [44].

The frequency domain analysis begins by evaluating the output response of a nonlinear system to an input $u(t)$ composed of K sinusoids with different frequencies and phase shifts:

$$u(t) = \sum_{i=1}^{K} |A_i| \cos(\omega_i t + \angle A_i) = \sum_{\substack{i=-K \\ i\neq 0}}^{K} \frac{A_i}{2}\mathrm{e}^{\mathrm{j}\omega_i t}, \tag{29}$$

where ω_k is the kth frequency with amplitude $|A_k|$ and phase shift $\angle A_k$. A_k is a complex number which gives the amplitude and phase of the kth frequency with the property that $A_{-k} = A_k^*$, where A_k^* is the complex conjugate of A_k. The total response of the nonlinear system can be expressed as

$$y(t) = \sum_{n=1}^{N_l} y_n(t), \tag{30}$$

where $y_n(t)$, the nth-order output of the system is given by

$$y_n(t) = \frac{1}{2^n} \sum_{k_1=-K}^{K} \cdots \sum_{k_n=-K}^{K} (A_{k_1}, \ldots, A_{k_n} H_n(\mathrm{j}\omega_{k_1}, \ldots, \mathrm{j}\omega_{k_n}))\mathrm{e}^{\mathrm{j}(\omega_{k_1}+\cdots+\omega_{k_n})t}. \tag{31}$$

Thus, the output consists of all possible combinations of the input frequencies $-\omega_{-K}, \ldots, -\omega_{-1}, \omega_1, \ldots, \omega_K$ taken n-at a time. The expression for the output with a specific frequency component is derived by defining the nth-order module or frequency mix vector of the input $M = (m_{-K}, \ldots, m_{-1}, m_1, \ldots, m_K)$ where $m_k \geqslant 0$ and $\sum_{i=-K}^{K} m_i = n$. m_k denote the number of times the frequency $f_k = \omega_k/2\pi$ appear in the frequency mix. An arbitrary frequency mix is then represented by the vector as

$$f_M = \sum_{\substack{i=-K \\ i\neq 0}}^{K} m_i f_i = \sum_{i=1}^{K} (m_i - m_{-i}) f_i. \tag{32}$$

The sum of all terms with the frequency f_M in the nth-order output component $y_n(t)$ is given as

$$y_n(t; f_M) = \frac{n!}{2^n} \left[\prod_{\substack{i=-K \\ i\neq 0}}^{K} \frac{A_i^{m_i}}{m_i!} \right] H_n(m_{-K}\{f_{-K}\}, \ldots, m_1\{f_{-1}\}, m_1\{f_1\}, \ldots, m_K\{f_K\})\mathrm{e}^{\mathrm{j}2\pi f_M t}, \tag{33}$$

where $m_K\{f_K\}$ denotes m_K consecutive arguments with the same frequency f_K. Note that $y_n(t;f_M)$ is a complex phasor rather than a sinusoidal function and the real sinusoidal component at frequency f_M can be obtained from

$$\tilde{y}_n(t;f_M) = y_n(t;f_M) + y_n^*(t;f_M) = 2\,\mathrm{Re}\{y_n(t;f_M)\}. \tag{34}$$

4.1. Example 4: Duffing oscillator with retarded damping

The effects of delay on several properties of the nonlinear system will be studied by considering the Duffing oscillator with retarded damping given by

$$\ddot{y}(t) + k_1\dot{y}(t) + k_2\dot{y}(t-T) + c_1 y(t) + k_3 y^3(t) = bu(t). \tag{35}$$

Models such as in Eq. (35) have been used in the study of anti-rolling stabilisation systems in ships where an artificially produced damping term $k_2\dot{y}(t-T)$ is added to systems with insufficient natural damping $k_1\dot{y}(t)$ [45]. The first- and third-order FRFs of the system are

$$H_1(\mathrm{j}\omega_1) = \frac{b}{(\mathrm{j}\omega_1)^2 + k_1(\mathrm{j}\omega_1) + k_2(\mathrm{j}\omega_1)\mathrm{e}^{-\mathrm{j}\omega_1 T} + c_1}, \tag{36}$$

$$H_3(\mathrm{j}\omega_1,\mathrm{j}\omega_2,\mathrm{j}\omega_3) = \frac{-k_3 H_1(\mathrm{j}\omega_1)H_1(\mathrm{j}\omega_2)H_1(\mathrm{j}\omega_3)}{(\mathrm{j}\omega_1+\mathrm{j}\omega_2+\mathrm{j}\omega_3)^2 + (k_1 + k_2\mathrm{e}^{-(\mathrm{j}\omega_1+\mathrm{j}\omega_2+\mathrm{j}\omega_3)T})(\mathrm{j}\omega_1+\mathrm{j}\omega_2+\mathrm{j}\omega_3) + c_1}, \tag{37}$$

respectively. The simulation is carried out with $k_1 = 0.2$, $k_2 = 0.16$, $c_1 = 1.0$, $k_3 = 1$ and $b = 1$. The plot of the first- and third-order GFRFs at different values of delay are shown in Fig. 1. From the plot of linear FRFs, it

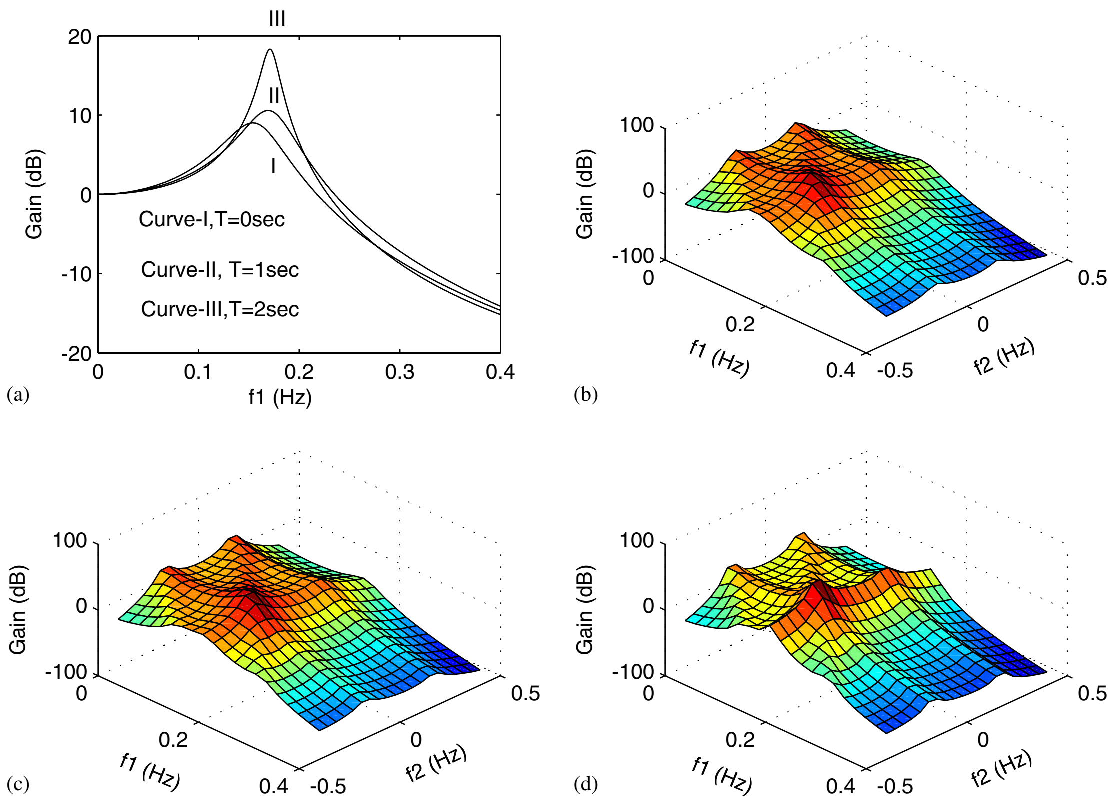

Fig. 1. First- and third-order frequency response functions of Duffing Oscillator with retarded damping at different values of delay T.

is observed that the peak value of the linear gain increases with increase in the delay T, and the oscillator become more selective with increasing T. The maximum linear gain at $T = 0$, 1 and 2 s are 9.0158, 10.5766 and 18.3022 dB, respectively. The peak value of third-order FRFs at $T = 0$, 1 and 2 s are 35.2485, 37.7342 and 56.2368 dB, respectively. This is expected since the higher-order FRFs depend on lower-order FRFs.

4.1.1. *Effects of delay in nonlinear terms*

In order to investigate the effects of the presence of delay in the nonlinear term, consider the Duffing oscillator of Eq. (35) with delay in the nonlinear term. Without loss of generality, the delay is assumed to be equal in both linear and nonlinear terms. Eq. (35) therefore changes to

$$\ddot{y}(t) + k_1\dot{y}(t) + k_2\dot{y}(t-T) + c_1 y(t) + k_3 y^3(t-T) = bu(t). \tag{38}$$

Inclusion of the delay in nonlinear terms will not affect the first-order FRF. However, the third-order FRF changes from Eq. (37) to

$$H_3(\mathrm{j}\omega_1, \mathrm{j}\omega_2, \mathrm{j}\omega_3) = \frac{-k_3 H_1(\mathrm{j}\omega_1) H_1(\mathrm{j}\omega_2) H_1(\mathrm{j}\omega_3) \mathrm{e}^{-\mathrm{j}(\omega_1+\omega_2+\omega_3)T}}{(\mathrm{j}\omega_1 + \mathrm{j}\omega_2 + \mathrm{j}\omega_3)^2 + (k_1 + k_2 \mathrm{e}^{-(\mathrm{j}\omega_1+\mathrm{j}\omega_2+\mathrm{j}\omega_3)T})(\mathrm{j}\omega_1 + \mathrm{j}\omega_2 + \mathrm{j}\omega_3) + c_1}. \tag{39}$$

By comparing Eq. (37) with Eq. (39) it is obvious that the presence of delay in nonlinear term will not affect the magnitude of the higher-order FRFs since the magnitude of $\mathrm{e}^{-\mathrm{j}(\omega_1+\omega_2+\omega_3)T}$ is unity. However, this will alter the phase response of various orders of higher-order FRF and may be analysed following the work of Zhang and Billings [46]. However, as will be evident from subsequent sections, the effects of delay on gain compression, desensitisation, etc. depend on the magnitude of the FRFs, and therefore the analysis of the effects of delay will be carried out by considering the system of Eq. (35) that contains delay in the linear terms.

4.2. *Effect of delay on harmonics*

When the system of Eq. (30) is excited by a single frequency sinusoid of frequency f harmonics are generated. It is possible to show that odd harmonics are, generated by odd order FRFs, and even harmonics are, generated by all even order FRFs [11]. The output of the system with frequency lf is given by

$$y(t; lf) = \mathrm{Re}\left\{ A^l \mathrm{e}^{\mathrm{j}2\pi(l)ft} \left[\frac{1}{2^{l-1}} H_l(l\{f\}) + \frac{l+2}{2^{l+1}} |A|^2 H_{l+2}(-f, (l+1)\{f\}) + \frac{(l+4)(l+3)}{2^{(l+4)}} |A|^4 H_{l+4}(-f, -f, (l+2)\{f\}) + \cdots \right] \right\}. \tag{40}$$

The magnitude of the lth harmonic is

$$|E_l| = |A|^l \left| \frac{1}{2^{(l-1)}} H_l(l\{f\}) + \frac{(l+2)}{2^{(l+1)}} |A|^2 H_{l+2}(-f, (l+1)\{f\}) + \cdots \right|. \tag{41}$$

Thus, the magnitude of the lth harmonic is proportional to the lth power of the input amplitude. If the amplitude of the input is taken to be smaller than one, then $|E_l|$ will predominantly dependent on the first term to give

$$|E_l| \simeq \frac{|A|^l}{2^{l-1}} |H_l(l\{f\})|. \tag{42}$$

In order to study the effect of delay on the harmonic generation, the Duffing oscillator represented by Eq. (35) was excited by the input

$$u(t) = A\cos 2\pi f t, \quad \text{where } A = 0.1 \text{ and } f = 0.1. \tag{43}$$

The magnitudes of the third harmonic at different values of delay T are normalised by dividing these with $|E_l|$ evaluated at $T = 0$. The normalised amplitude of the lth harmonic

$$|E_l|(\text{Normalised}) = \frac{|E_l|}{|E_l|_{at\ T=0}}. \tag{44}$$

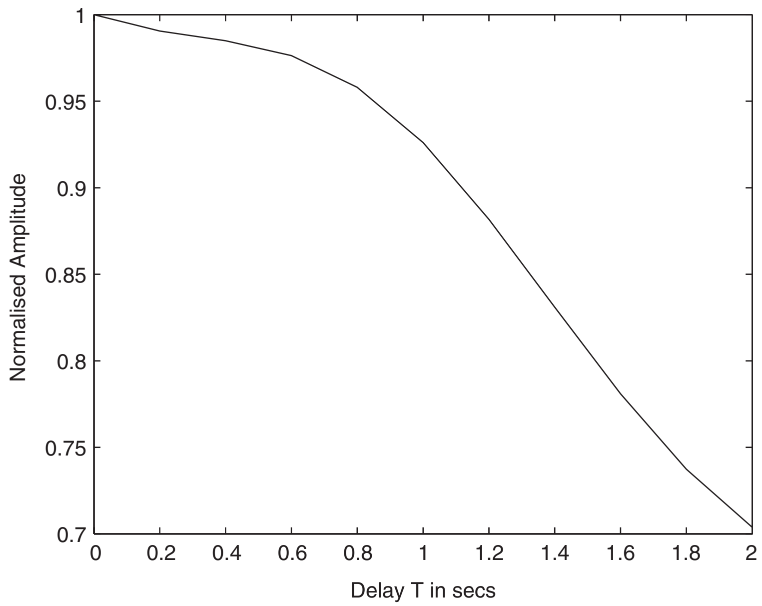

Fig. 2. Normalised amplitude of third harmonic at different values of delay T.

From the plot it is observed that the amplitude of the third harmonic due to the sinusoidal excitation of frequency 0.1 Hz gradually decrease with increase in delay. Note that the magnitude of the lth harmonic is dependent on the magnitude of lth-order FRF $H_l(\cdot)$ evaluated along the line $f_1 = f_2 = \cdots = f_l$. Therefore, the magnitude of $H_l(\cdot)$ at the excitation frequency essentially determines the amplitude of lth harmonic (Fig. 2).

4.3. *Effect of delay on gain compression/expansion*

The gain of a nonlinear system usually depends on the amplitude of the input signal. The output increases linearly with the input amplitude up to a certain point as would be the case in a linear system, and then fails to follow a linear relationship. The linear relationship between the output amplitude and input holds good for a limited range of input amplitude. The effect of delay on the gain compression/ expansion is studied by exciting the system with a single sinusoid of frequency f and amplitude $|A|$. The output of the system at the fundamental frequency is given by [11,39,47,48].

$$y(t;f) = \mathrm{Re}\{A\mathrm{e}^{\mathrm{j}2\pi ft}[H_1(f) + \tfrac{3}{4}|A|^2 H_3(-f,f,f) + \tfrac{5}{8}|A|^4 H_5(-f,-f,f,f,f) + \cdots]\}. \tag{45}$$

The gain of the system at the fundamental frequency f is defined by the describing function $N(A,f)$ given as

$$|N(A,f)| = |H_1(f) + \tfrac{3}{4}|A|^2 H_3(-f,f,f) + \tfrac{5}{8}H_5(-f,-f,f,f,f) + \cdots|. \tag{46}$$

In case of linear systems, all the higher-order FRFs above order 1 are zero, and the gain function $|N(A,f)|$ is independent of the input amplitude and equal to the linear FRF evaluated at frequency f. However, nonlinear systems exhibit a behaviour that differs increasingly from the linear systems as the input amplitude and the magnitude of the higher-order GFRFs change. If the nonlinear effects above third order are negligible, then the input–output relation can approximately be expressed as

$$\begin{aligned} y &= \left| H_1(f) + \frac{3}{4}A^2 H_3(-f,f,f) \right| \\ &\approx |H_1(f)| \underbrace{\left| 1 + \frac{3}{4}A^2 \frac{H_3(-f,f,f)}{H_1(f)} \right|} A. \end{aligned} \tag{47}$$

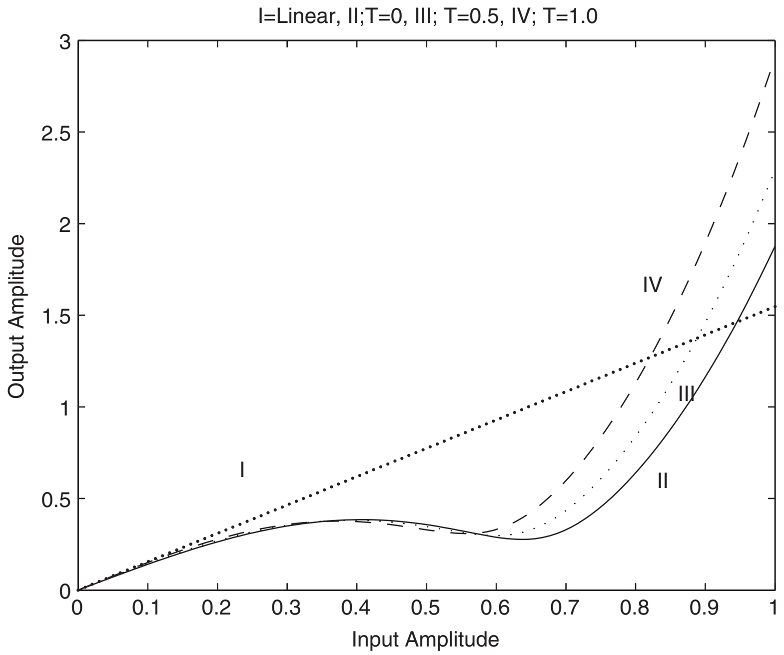

Fig. 3. Illustrating effects of delay on gain compression/expansion: output amplitude at fundamental frequency vs. input amplitude at constant frequency for Duffing oscillator with retarded damping at various values of delay T.

The factor in underbrace in Eq. (47) represents the gain compression or expansion of the nonlinear systems with respect to the first-order FRF $H_1(f)$. Since the delay in the output affects the magnitude of all the FRFs, it is obvious that the delay will have significant effect on the gain compression/expansion phenomena. In order to study this phenomena, the system of Eq. (35) excited by a sinusoid of frequency 1 Hz. The variations of output amplitudes with input amplitude are shown in Fig. 3 for different values of delay T. From the figure it is observed that the system's behaviour departs from the linear one as the amplitude is increased with gain compression for small input amplitudes and gain expansion for medium and high input amplitudes. Moreover, for the choice of the parameter in the simulation, the delay has less effect at low ranges of the input amplitude. The effect of delay has been significant in medium and high ranges of input amplitudes.

4.4. *Effect of delay on desensitisation*

For a linear system, the sinusoidal response at a frequency f_1 is unaffected by exciting the system with another frequency f_2. However, for a nonlinear system, the response at a given frequency f_1 is affected by the application of another sinusoid of different frequency f_2. This phenomenon is called *desensitisation* [44] and is studied in this section by exciting the Duffing oscillator with the input

$$u(t) = |A_1| \cos(2\pi f_1 t + \angle A_1) + |A_2| \cos(2\pi f_2 t + \angle A_2). \tag{48}$$

Since the excitation consists of a two-tone input, the input frequencies to the system are $-f_2, -f_1, f_1$ and f_2.

The nth-order module vector of the input frequencies will take the form $M = (m_{-2}, m_{-1}, m_1, m_2)$. The output at frequency f_1 may be expressed as

$$y(t; f_1) = \mathrm{Re}\{[A_1 H_1(f_1) + \tfrac{3}{4} A_1 |A_1|^2 H_3(-f_1, f_1, f_1) + \tfrac{3}{2} A_1 |A_2|^2 H_3(-f_2, f_1, f_2) + \cdots] \mathrm{e}^{\mathrm{j}2\pi f_1 t}\}. \tag{49}$$

Since the intention is to study the effects of delay on desensitisation, $|A_1|$ is assumed to be very much less than $|A_2|$ to keep the effect of *gain compression/expansion* very small compared to the desensitisation term.

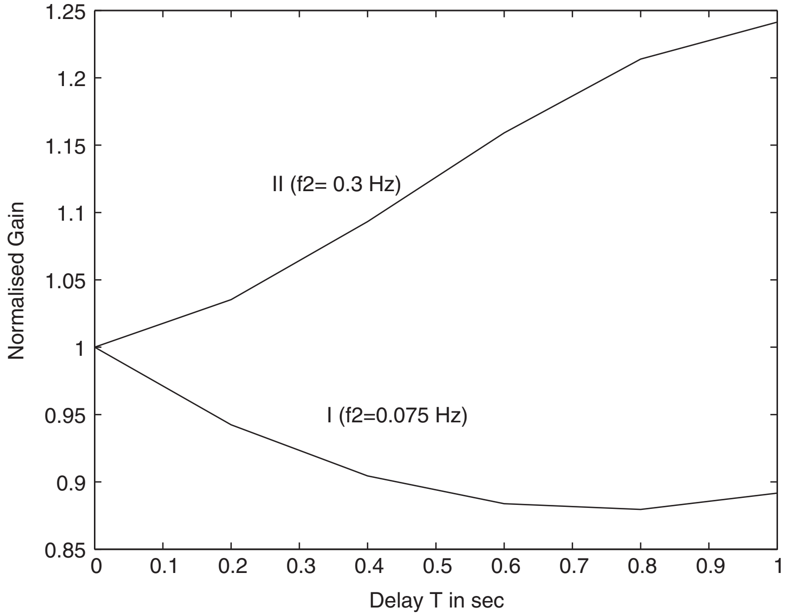

Fig. 4. Normalised gain of system at frequency f_1 at different values of delay T.

If the nonlinear effects above third order are neglected

$$y(t;f_1) = \mathrm{Re}\{[A_1 H_1(f_1) + \tfrac{3}{2} A_1 |A_2|^2 H_3(-f_2, f_1, f_2)]\mathrm{e}^{\mathrm{j}2\pi f_1 t}\}. \tag{50}$$

The amplitude of the output signal at frequency f_1 becomes

$$\mathrm{Gain}[\, y(t;f_1)] = |A_1 H_1(f_1) + \tfrac{3}{2} A_1 |A_2|^2 H_3(-f_2, f_1, f_2)|. \tag{51}$$

The gain of the system at frequency f_1 is given by

$$S_{\mathrm{gain}} = \frac{\mathrm{gain}\{y(t : f_1)\}}{|A_1|} = |H_1(f_1)| \left| 1 + \frac{3}{2} |A_2|^2 \frac{H_3(-f_2, f_1, f_2)}{H_1(f_1)} \right|. \tag{52}$$

To study the interfering effect of the signal at frequency f_2, system (35) was excited by the input

$$u(t) = A_1 \cos 2\pi f_1 t + A_2 \cos 2\pi f_2 t \tag{53}$$

with $A_1 = 0.1$, $A_2 = 3.0$ and $f_1 = 0.15\,\mathrm{Hz}$. The effect of delay on desensitisation was initially studied by applying the second signal with $f_2 = 0.3\,\mathrm{Hz}$. The normalised value of the system gain at frequency f_1, which is the ratio of system gain at frequency f_1 to the system gain at frequency f_1 with no delay, is plotted in Fig. 4(I). The plot shows that the effect of the second signal on the gain of the system at f_1 increases with delay. Note that $S_{\mathrm{gain}}(f_1)$ is a nonlinear function of the amplitude and frequency of the interfering signal. The effect of delay on the gain can be different if frequency of the second signal changes to another value. To demonstrate the highly nonlinear nature of $S_{\mathrm{gain}}(f_1)$, the frequency of the second signal f_2 was changed from 0.3 to 0.075 Hz. The variation of $S_{\mathrm{gain}}(f_1)$ with delay for $f_2 = 0.075\,\mathrm{Hz}$ is shown in Fig. 4(II). From the figure, it is observed that the gain decreases with delay, unlike the case when $f_2 = 0.3\,\mathrm{Hz}$.

5. Conclusions

Algebraic expressions in a recursive framework have been derived for GFRFs of nonlinear time delay systems using the operator ε_n. The expressions give an important insight into the relationships between time and frequency domain representations of the system, and can be used to extract frequency domain information about the system. The GFRFs derived have been used to demonstrate the effects of delay on nonlinear phenomena of harmonics, gain compression and expansion and desensitisation by considering the example of a nonlinear Duffing oscillator with retarded damping. Finite-dimensional discrete time models can be reconstructed from the GFRFs of delay systems that are essentially of infinite dimensional and will be the part of future research. Moreover, the stability of the nonlinear delay systems can be studied based on the GFRFs.

References

[1] S.A. Billings, Identification of nonlinear systems—a survey, *Proceedings of the IEEE—Part D* 127 (1980) 272–285.
[2] M. Schetzen, *The Volterra and Wiener Theories of Nonlinear Systems*, Wiley, New York, 1980.
[3] M.J. Korenberg, S.B. Bruder, P.J. Mcllroy, Exact orthogonal kernel estimation from finite data records: extending Wiener identification of nonlinear systems, *Annals of Biomedical Engineering* 16 (1988) 201–214.
[4] M.J. Korenberg, I.W. Hunter, The identification of nonlinear biological systems: Wiener approaches, *Annals of Biomedical Engineering* 18 (1990) 629–654.
[5] G. Hung, L. Stark, The interpretation of kernels—an overview, *Annals of Biomedical Engineering* 19 (1991) 509–519.
[6] D.T. Westwick, R.E. Kearney, A new algorithm for identification of multi-input wiener systems, *Biological Cybernetics* 68 (1992) 75–85.
[7] K.I. Kim, E.J. Powers, A digital method of modelling quadratically nonlinear systems with a general random input, *IEEE Transactions of the ASSP* 36 (1988) 1758–1769.
[8] Y.C. Tseng, D.A. Lineberger, Estimating transfer functions of quadratically nonlinear systems, in: *Proceedings of 25th Aslimore Conference*, 1991, pp. 632–636.
[9] Y.C. Tseng, E.J. Powers, Identification of cubic systems using higher order moments of i.i.d signals, *IEEE Transactions of Signal Processing* 43 (7) (1995) 1733–1735.
[10] E. Bedrosian, S.O. Rice, The output properties of Volterra systems (nonlinear systems with memory) driven by harmonic and Gaussian inputsory, *Proceedings of IEEE* 59 (1971) 1688–1707.
[11] S.A. Billings, K.M. Tsang, Spectral analysis for nonlinear systems, Part I: parametric nonlinear spectral analysis, *Mechanical Systems and Signal Processing* 3 (4) (1989) 319–339.
[12] J.C. Peyton-Jones, S.A. Billings, Recursive algorithms for computing the frequency response of a class of non-linear difference equation models, *International Journal of Control* 50 (1989) 1925–1940.
[13] H. Zhang, S.A. Billings, Q.M. Zhu, Frequency response functions for nonlinear rational models, *International Journal of Control* 61 (1995) 1073–1097.
[14] A.K. Swain, S.A. Billings, Generalised transfer function matrix for mimo nonlinear systems, *International Journal of Control* 74 (8) (2001) 829–844.
[15] I.J. Leontaritis, S.A. Billings, Input–output parametric models for nonlinear systems, Part I: deterministic nonlinear systems, *International Journal of Control* 41 (2) (1985) 303–328.
[16] M. Zaveri, M. Jamshidi, *Time Delay Systems: Analysis, Optimisation and Applications*, North-Holland, Amsterdam, 1987.
[17] R. Bellman, K.L. Cooke, *Differential-difference Equations*, Academic Press, New York, 1963.
[18] A. Manitius, Optimal control of heriditary systems, control theory and topics in functional analysis, *International Atomic Energy Agency* 3 (1976) 43–178.
[19] J.K. Hale, *Theory of Functional*, Springer, Berlin, 1977.
[20] R.D. Driver, *Ordinary and Delay Differential Equations*, Springer, Berlin, 1977.
[21] L. Dugard, E.I. Verriest (Eds.), *Stability and Control of Time-delay Systems*, Springer, Berlin, Germany, 1997.
[22] J. Hale, S.M.V. Lunel, *Introduction to Functional Differential Equations*, Springer, New York, 1993.
[23] R.M. Palhares, C.E. de Souza, P.L.D. Peres, Robust $\mathscr{H}_\infty$ filtering for uncertain discrete-time state-delayed system, *IEEE Transactions on Signal Processing* 49 (8) (2001) 1696–1703.
[24] R.M. Palhares, C.D. Campos, P.Y. Ekel, M.C.R. Leles, M.F.S.V. D'Angelo, Delay-dependent robust $\mathscr{H}_\infty$ control of uncertain linear systems with lumped delays, *IEE Proceedings—Control, Theory and Applications* 152 (8) (2005) 27–33.
[25] R.M. Palhares, C.D. Campos, P.Y. Ekel, M.C.R. Leles, M.F.S.V. D'Angelo, Delay-dependent robust $\mathscr{H}_\infty$ control of uncertain linear systems with time-varying delays, *Computers & Mathematics with Applications* 50 (1–2) (2005) 13–32.
[26] S.-L. Niculescu, K. Gu (Eds.), *Advances in Time-delay Systems*, Springer, Berlin, 2004.
[27] E.N. Chukwu, *Stability and Time-optimal Control of Heriditary Systems*, Academic Press, New York, 1992.
[28] B. Nicolaenko, C. Foias, R. Temam (Eds.), *The Connection Between Infinite Dimensional and Finite Dimensional Dynamical Systems: Proceedings of the AMS–IMS–SIAM Joint Summer Research Conference*, July 19–25, 1987, American Mathematical Society, Providence, RI, 1997.
[29] V. Volterra, *Theory of Functionals and of Integral and Integro-Differential Equations*, Dover, New York, 1959.
[30] S.A. Billings, J.C. Peyton-Jones, Mapping nonlinear integro-differential equations into the frequency domain, *International Journal of Control* 52 (1990) 863–879.
[31] K. Worden, G. Manson, G.R. Tomlinson, A harmonic probing algorithm for the multi-input Volterra series, *Journal of Sound and Vibration* 201 (1) (2001) 67–84.

[32] J.F. Barrett, The use of Volterra series to find regions of stability of a nonlinear differential equation, *International Journal of Control* 1 (1965) 209–316.
[33] G.R. Tomlinson, G. Manson, G.M. Lee, A simple criterion for establishing an upper limit of the harmonic excitation level to the Duffing oscillator using the Volterra series, *Journal of Sound and Vibration* 190 (1996) 751–762.
[34] S.A. Billings, Z.Q. Lang, A bound on the magnitude characteristics of nonlinear output frequency response functions, Part I: analysis and computation, *International Journal of Control* 65 (2) (1996) 309–328.
[35] G. Palm, T. Poggio, The Volterra representation and the Wiener expansion: validity and pit falls, *SIAM Journal on Applied Mathematics* 33 (2) (1977) 195–216.
[36] G.F. Simmons, *Introduction to Topology and Modern Analysis*, McGraw-Hill, New York, 1963.
[37] H.T. Banks, J.A. Burns, Heriditary control problems: numerical methods based on averaging approximations, *SIAM Journal of Control and Optimisation* 16 (2) (1978) 169–208.
[38] L.A. Aguirre, S.A. Billings, Improved structure selection based on term clustering, *International Journal of Control* 62 (3) (1995) 59–87.
[39] J.C. Peyton Jones, S.A. Billings, Describing functions, Volterra series, and the analysis of nonlinear systems in the frequency domain, *International Journal of Control* 53 (4) (1991) 871–887.
[40] H. Zhang, S.A. Billings, Analysing nonlinear systems in the frequency domain, I: the transfer function, *Mechanical Systems and Signal Processing* 7 (1993) 531–550.
[41] K. Worden, G.R. Tomlinson, *Nonlinearity in Structural Dynamics, Detection, Identification and Modelling*, Institute of Physics Publishing, Bristol, Philadelphia, 2001.
[42] J.A. Feijoo Vazquez, K. Worden, R. Stanway, System identification using associated linear equations, *Mechanical Systems and Signal Processing* 18 (2004) 431–455.
[43] J.A. Feijoo Vazquez, K. Worden, R. Stanway, Associated linear equations for Volterra operators, *Mechanical Systems and Signal Processing* 19 (1) (2005) 57–69.
[44] D.D. Wiener, D.F. Spina, *Sinusoidal Analysis and Modelling of Weakly Nonlinear Circuits*, Van Nostrand, New York, 1980.
[45] N. Minorsky, *Nonlinear Oscillations*, Van Nostrand, New York, 1962.
[46] H. Zhang, S.A. Billings, Analysing nonlinear systems in the frequency domain, II: the phase response, *Mechanical Systems and Signal Processing* 8 (1) (1994) 45–62.
[47] J.C. Peyton-Jones, Mapping nonlinear integro-differential equations to a generalised describing function form, *International Journal of Control* 62 (1995) 1145–1160.
[48] J.C. Peyton-Jones, Polyharmonic balance analysis of nonlinear ship roll response, *Nonlinear Dynamics* 35 (2) (2004) 123–146.